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# DIMENSION ESTIMATE OF POLYNOMIAL GROWTH HARMONIC FORMS

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## Abstract

Let $H_l^p(M)$ be the space of polynomial growth harmonic forms. We proved that the dimension of such spaces must be finite and can be estimated if the metric is uniformly equivalent to one with a nonnegative curvature operator. In particular, this implies that the space of harmonic forms of fixed growth order on the Euclidean space with any periodic metric must be finite dimensional.


## 1. Introduction

The classical de Rham-Hodge theory implies that the dimension of the space of harmonic forms is a topological invariant of a compact Riemannian manifold, hence independent of the choice of the background Riemannian metric. For the complete noncompact manifolds, this is no longer true. Nonetheless, it is an intriguing question to study the space of harmonic forms and to seek for topological and geometrical links. In the case of harmonic functions, Yau [**12**] has proved that any positive harmonic function on a manifold with nonnegative Ricci curvature must be constant. Hence the strong Liouville property holds. Later, Saloff-Coste [**11**] showed this still holds true under a uniformly equivalent metric. So the space of positive harmonic functions is stable under a quasi-isometry for such manifolds.

A complete manifold $M$ is said to satisfy a Sobolev inequality $S(A,\nu)$ if there exist a point $q \in M$ and constants $A > 0$ and $\nu > 2$ such that for all $r > 0$, and for all $f \in C_0^\infty(B_q(r))$, we have

$$\int_{B_q(r)} |f|^{\frac{2\nu}{\nu-2}} \leq A r^2 V(q,r)^{-\frac{2}{\nu}} \int_{B_q(r)} (|\nabla f|^2 + r^{-2} f^2)$$

where $V(q,r)$ is the volume of the geodesic ball $B_q(r)$. Examples of manifolds satisfying such a Sobolev inequality include minimal submanifolds with Euclidean volume growth in $\mathbb{R}^m$ and manifolds with a nonnegative Ricci curvature. It is also obvious that the Sobolev inequality holds

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true, possibly with a different constant $A$, under any uniformly equivalent metric on $M$.

In [**10**], extending earlier work of Li [**8**], Li and Wang proved that the dimension of the space $H_l^0(M)$ of polynomial growth harmonic functions of growth order at most $l$ has an estimate

$$\dim H_l^0(M) \leq C(A,\nu) l^\nu$$

provided that the underlying manifold satisfies the Sobolev inequality $S(A,\nu)$. So the finite dimensionality of the space $H_l^0(M)$ is valid on such a manifold with respect to any uniformly equivalent metric.

Concerning the general harmonic $p$-forms, Li [**8**] has also established a dimension estimate of the space of polynomial growth harmonic forms. Assume that $K_p \geq 0$ on $M^m$, where

$$K_P = \begin{cases} \text{lower bound of the curvature operator on } M & \text{if } p > 1, \\ (m-1)^{-1} \times (\text{lower bound} & \\ \qquad \text{of the Ricci curvature Ricci curvature}) & \text{if } p = 1. \end{cases}$$

Then his result says that

$$\dim H_l^p(M) \leq C l^{m-1},$$

where $H_l^p(M)$ denotes the space of polynomial growth harmonic $p$-forms on $M$ of growth order at most $l$.

In view of the preceding results on harmonic functions, it is natural to ask if the dimension of the space $H_l^p(M)$ is stable under the change of the metric to a uniformly equivalent one. If so, then it becomes possible to relate the dimension to certain topological invariants of the manifold. The goal of this paper is to show that the dimension remains finite.

**Theorem 1.1.** *Let $(M^m, g)$ be a complete Riemannian manifold with dimension $m$. Suppose that $K_p \geq 0$ on $(M,g)$ and the metric $g'$ is uniformly equivalent to $g$ on $M$. Then there exist constants $C > 0$ and $\nu > 2$ such that the dimension of the space $H_l^p(M,g')$ is finite and satisfies the inequality*

$$\dim H_l^p(M,g') \leq C\, l^\nu$$

*for all $l \geq 1$.*

An immediate corollary is that on the Euclidean space equipped with any periodic metric the space of polynomial growth harmonic forms of fixed growth order must be finite dimensional.

**Corollary 1.2.** *Let $g$ be a periodic Riemannian metric on $\mathbb{R}^m$. Then*

$$\dim H_l^p(\mathbb{R}^m, g) \leq C\, l^m$$

*for all $p \geq 1$ and $l \geq 1$.*

Note that this corollary holds true for the case $p = 0$ by the result of Avellaneda and Lin [**1**]. In fact, they have proven a much stronger result by giving an explicit description of the polynomial growth harmonic functions and establishing a one to one correspondence with the ones with respect to the standard metric on $\mathbb{R}^m$. It remains an interesting question whether a similar description is also available for the harmonic forms.

The proof of the aforementioned result of Li [**8**] relies on the Bochner-Weitzenbock formula. Under a uniformly equivalent change of the metric, however, the same curvature condition is not expected to hold. Therefore, we need to take a different approach to estimate the dimension. We overcome the difficulty by relating the space of harmonic forms to the eigenvalues of the Hodge Laplacian on the geodesic balls with respect to the absolute boundary conditions. This kind of idea was first introduced and successfully pursued by Li and Wang [**10**] for the harmonic functions. Here, the added point is to show that such eigenvalues are comparable under a uniformly equivalent change of the metric. This of course can be easily seen from the variational characterization of the eigenvalues for the function case. In the case of general forms, some effort is required to show such a fact. So we will first study the eigenvalues in section 2 and obtain a lower bound estimate by modifying an argument from [**7**]. The main result is then proved in section 3.

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## 2. Eigenvalue estimates

Let $(M^m, g)$ be a complete, oriented Riemannian manifold. The Hodge-Laplace-Beltrami operator $\Delta$ acting on the space of smooth $p$-forms $\Lambda^p(M)$ is defined as

$$\Delta = d\delta + \delta d,$$

where $d$ denotes the exterior differential operator and $\delta = * \, d \, *$, where the linear operator $*$ is defined point-wise by

$$*(w_1 \wedge \cdots \wedge w_p) = w_{p+1} \wedge \cdots \wedge w_m$$

for a positively oriented orthonormal co-frame $\{w_1,\ w_2, \ldots,\ w_m\}$ at the point. A $p$-form $w \in \Lambda^p(M)$ is called a harmonic $p$-form on $(M, g)$ if

$$\Delta_g w = 0.$$

Let $q$ denote a point on $(M, g)$ and let $r_q(x)$ represent the geodesic distance function from $x \in M$ to the point $q$. For each $l \geq 0$, we denote the space of polynomial growth harmonic $p$-forms of degree at most $l$ by

$$H_l^p(M,g) \equiv \Big\{ w \in \Lambda^p(M) \mid \Delta_g w = 0, \text{ and } |w| = O(r_q^l) \Big\}.$$

For a bounded smooth domain $B \subset M$, a $p$-form $w$ is said to satisfy the absolute condition on $B$ if the tangential component of both $w$ and $\delta w$ on the boundary $\partial B$ are zeros. On the boundary $\partial B$, let $N_{\partial B}$ (respectively $N^*_{\partial B_q}$) represent the inward unit normal vector (respectively co-vector) field. Now, denote exterior multiplication by $\operatorname{ext}(\cdot)$ and dual exterior multiplication by $\operatorname{int}(\cdot)$. It is not difficult to verify that $\Delta$ is a self-adjoint nonnegative operator on the space $\Lambda^p(B)$ of smooth $p$-forms on $B$ satisfying the absolute boundary conditions. By the standard elliptic theory, we see that $\Delta$ has a countable set of eigenvalues and the multiplicity of each eigenvalue is finite. If we list all the eigenvalues with multiplicity in nondecreasing order by $\{\lambda_k,\ k = 1, 2, 3, \dots\}$, then $\lambda_k \to \infty$ as $k \to \infty$. Moreover, the $i$-th eigenvalue can be characterized as

$$\lambda_i = \inf_{\dim V = i} \sup_{w \in V \setminus \{0\}} R(w),$$

where $V$ is a subspace of $\Lambda^p(B)$ and the Rayleigh-Ritz quotient $R(w)$ is defined by

$$R(w) = \frac{(dw, dw) + (\delta w, \delta w)}{(w, w)}$$

for $w \in \Lambda^p(B)$ and the $L^2$ inner product for two forms $v$ and $w$ in $\Lambda^p(B)$ is defined by

$$(v, w) = \int_B \langle v, w \rangle dx$$

with $\langle v, w \rangle$ being the point-wise inner product between $v$ and $w$.

On the other hand, the Hodge-de Rham theorem provides an orthogonal decomposition of the space $\Lambda^p(B)$ of differential forms of degree $p$ on $B$. For any $w \in \Lambda^p(B)$, $w$ can be uniquely written as

$$w = h + dv + \delta\, u,$$

where $h \in H_p(B)$, the space of harmonic $p$-forms satisfying the absolute boundary conditions and $v \in \Lambda^{p-1}(B)$, $u \in \Lambda^{p+1}(B)$. Clearly, the operator $\Delta$ leaves this decomposition invariant, and the eigenvalues of $\Delta$ on the subspace $H_p(B)$ are zeros.

Denote by $\{\mu_j^e(g) | j \geq 1\}$ the eigenvalues of $\Delta$ acting on the subspace $d\Lambda^{p-1}(B)$ of exact $p$-forms, and by $\{\mu_l^{co}(g) | l \geq 1\}$ those corresponding to the subspace $\delta\Lambda^{p+1}(B)$ of co-exact $p$-forms. Then the eigenvalues $\{\lambda_i(g) | i > \dim H_p(B)\}$ is equal to the re-ordered union of $\{\mu_j^e(g) | j \geq 1\}$ and $\{\mu_l^{co}(g) | l \geq 1\}$. We have

$$\{\lambda_i(g) | i > \dim H_p(B)\} = \{\mu_j^e(g) | j \geq 1\} \cup \{\mu_l^{co}(g) | l \geq 1\}.$$

The following lemma is essentially due to [**3**]. We recall it here for our convenience.

**Lemma 2.1.** *Let $(M, g)$ be a complete manifold with Riemannian metric $g$. Let $g'$ denote another Riemannian metric on $M$ which is uniformly equivalent to $g$, that is, there exists a positive constant $C$ such that*

$$C^{-1}g \le g' \le Cg.$$

*Then*

(i) $\dim H_p(B, g) = \dim H_p(B, g')$

(ii) *There exists a positive constant $C$ such that*

$$C^{-1}\lambda_i(g) \le \lambda_i(g') \le C\lambda_i(g)$$

*for all $i \ge 1$.*

*Proof.* The first statement (i) is true as the space $H_p(B, g)$ can be identified with the $p$-th absolute cohomology of the set $B$ by the de Rham theorem.

To prove (ii), we first show the following claim.

$$\mu^e_{i,p}(g) = \inf_V \sup_{\phi \in V \setminus \{0\}} \left\{ \frac{(\phi, \phi)}{(\theta, \theta)} \,\middle|\, d\theta = \phi \right\},$$

where $V$ ranges over all subspaces of dimension $i$ of the space of all exact forms on $B$ satisfying the absolute boundary conditions.

The claim clearly implies that

$$C_1\mu^e_i(g) \le \mu^e_i(g') \le C_2\mu^e_i(g)$$

and

$$C_3\mu^{co}_i(g) \le \mu^{co}_i(g') \le C_4\mu^{co}_i(g).$$

Therefore, (ii) follows.

To prove the claim, denote the nonzero eigenvalue $\mu$ of the eigen $p$-form on $B$ with absolute boundary condition. In addition, let $E^p_d(\mu)$ (respectively $E^p_\delta(\mu)$) denote the space of all exact (respectively co-exact) eigen $p$-forms belonging to the eigenvalue $\mu$. Define a mapping $T$, where

$$T : E^p_d(\mu) \to E^{p-1}_\delta(\mu),$$

by

$$T\phi = \frac{1}{\mu}\delta\phi.$$

Denote the complete orthonormal set of exact eigen $p$-forms with respect to the absolute boundary conditions by $\{\phi_1, \phi_2, \phi_3, \ldots\}$ with the corresponding eigenvalues $0 < \mu^e_1 \le \mu^e_2 \le \mu^e_3 \le \cdots$ . Let

$$V_i \equiv \text{ span } \{\phi_1, \ldots, \phi_i\}.$$

Let $\Psi = T\phi$ for $\phi = \sum a_j\phi_j$. Then $\phi = d\Psi$ and

$$\frac{(\phi,\phi)}{(\Psi,\Psi)} = \frac{\sum a_j^2}{\sum(\mu_j^e)^{-1}a_j^2}.$$

Since $\mu_i^e = \max \mu_j^e$, we see that

$$\mu_i^e = \sup\left\{\frac{(\phi,\phi)}{(\Psi,\Psi)}\;\middle|\; \phi \in V_i\backslash\{0\}\right\}.$$

If $\phi = d\theta$, then

$$\theta = \Psi + d\Phi + \hat{H},$$

for some $(p-2)$-form $\Phi$ and $(p-1)$-harmonic-form $\hat{H}$ under the absolute boundary condition. This follows from the Hodge decomposition. Given a $p$-form $\overline{\Psi}$ such that $\Psi = \delta\overline{\Psi}$, we have

$$\begin{aligned}
(\Psi, d\Phi) &= (\delta\Psi,\Phi) - (\Psi, \operatorname{ext}(N^*)\Phi)_{L^2(\partial B)} = 0,\\
(\hat{H}, d\Phi) &= (\delta\hat{H},\Phi) - (\hat{H}, \operatorname{ext}(N^*)\Phi)_{L^2(\partial B)} = 0,\\
(\hat{H}, \delta\overline{\Psi}) &= (d\hat{H},\overline{\Psi}) + (\hat{H}, \operatorname{ext}(N^*)\overline{\Psi})_{L^2(\partial B)} = 0.
\end{aligned}$$

Thus, for a fixed $\phi$, we have

$$(\theta,\theta) = (\Psi,\Psi) + (\phi,\phi) + (\hat{H},\hat{H}).$$

It follows that

$$\frac{(\phi,\phi)}{(\Psi,\Psi)} = \sup\left\{\frac{(\phi,\phi)}{(\theta,\theta)}\;\middle|\; d\theta = \phi\right\},$$

then it is easy to see that

$$\mu_i^e = \sup\left\{\frac{(\phi,\phi)}{(\theta,\theta)}\;\middle|\; \phi \in V_i\backslash\{0\} \text{ and } d\theta = \phi\right\}.$$

So

$$\mu_{i,p}^e(g) \geq \inf_V \sup_{\phi\in V\backslash\{0\}} \left\{\frac{(\phi,\phi)}{(\theta,\theta)}\;\middle|\; d\theta = \phi\right\}.$$

To prove the reverse inequality, we first note that

$$\mu_{i,p}^e = \mu_{i,p-1}^{co}$$

as the operator $\delta$ commutes with the Laplacian $\Delta$. For any such a subspace $V = \{\phi_j\}_{j=1}^i$ of dimension $i$, $\phi_j$ is an eigen $p$-form, there exists a nonzero $\phi \in V$ so that $\phi$ is perpendicular to $V_{i-1}$ of dimension $(i-1)$ and $V_{i-1} \subset V$. We may write

$$\phi = d\theta$$

and

$$\phi_j = d\theta_j$$

for all $j$, where $\theta$ and $\theta_j$ are $(p-1)$-forms satisfying the absolute boundary conditions. Note that

$$\theta_j = \mu_j^{-1}\,\delta\,\phi_j, \quad \text{where } \ \mu_j \text{ is the eigenvalue of eigen } p\text{-form } \ \phi_j.$$

Hence,

$$(\theta, \theta_j) = 0 \quad \text{for all} \quad 1 \le j \le i-1.$$

Also, using the Hodge decomposition, we may assume $\theta$ is co-exact. Therefore,

$$\mu^e_{i,p}(\theta, \theta) = \mu^{co}_{i,p-1}(\theta, \theta) \le (d\theta, d\theta) = (\phi, \phi).$$

This shows that

$$\mu^e_{i,p}(g) \le \sup_{\phi \in V \setminus \{0\}} \left\{ \frac{(\phi, \phi)}{(\theta, \theta)} \,\middle|\, d\theta = \phi \right\}.$$

Since $V$ is arbitrary, this proves the reverse inequality and the claim. q.e.d.

In the following, we will get a lower bound estimate of the eigenvalues of $p$-forms satisfying the absolute boundary conditions on a geodesic ball in a manifold with $K_p \ge 0$. The argument closely follows those in [**7**].

**Proposition 2.2.** *Let $M^m$ be a complete manifold. Let $B = B_q(r)$ be a geodesic ball of radius $r > 0$. Then there exist constants $C > 0$ and $\nu > 2$ such that*

(i) $\dim H_p(B) \le C e^{C\sqrt{K_p} r}$,

(ii) *for* $k > \dim H_p(B)$,

$$\lambda_k(B) \ge C^{-1} \, k^{2/\nu} \, r^{-2} \, e^{-C\sqrt{K_p}\, r} + C\, K_p.$$

*Proof.* Let $E$ be the $k$-dimensional space spanned by the eigen $p$-forms corresponding to the first $k$ eigenvalues $\{\lambda_1, \dots, \lambda_k\}$ on $B$. Then there exists $w \in E$, $w \ne 0$, such that

$$\frac{k}{V} \, \|w\|_2^2 \le \|w\|_\infty^2 \cdot \min\{\tbinom{m}{p},\ k\}, \tag{1}$$

where $V$ denotes the volume of $B$. This is a result in [**7**].

On the other hand, we claim that there exist constants $C > 0$, $k_0 > 0$ and $\nu > 2$ such that for $w \in E$,

$$\|w\|_\infty^2 \le C(\nu) \left( e^{C(1+\sqrt{K_p} r)} V^{-2/\nu} r^2 (2\lambda_k - p(m-p)K_p) \right)^{\nu/2} \|w\|_2^2 \quad \text{for all} \quad k \ge k_0. \tag{2}$$

It is easy to see that the proposition follows by combining (1) and (2). To prove (2), we first observe that for $w \in \Lambda^p(B)$,

$$\frac{\partial |w|^2}{\partial n} = 0 \quad \text{on} \quad \partial B,$$

where $\frac{\partial}{\partial n}$ is the outward unit normal of $\partial B$. Indeed, for a point $x \in \partial B$, choose normal coordinates $\{x_1, \dots, x_m\}$ at $x$ so that $\{x_1, \dots, x_{m-1}\}$ forms a coordinate system on $\partial B$. So at $x$, we have

$$\frac{\partial}{\partial x_m} = \frac{\partial}{\partial n}.$$

Now write $w = \sum_I f_I dx_I$, where $I = \{i_1, \ldots, i_p\}$ with $i_1 < i_2 < \cdots < i_p$. Since $w_{tan} = 0$ on $\partial B$, we have $f_I = 0$ on $\partial B$ when $m \in I^c$. On the other hand, using this fact and also $(\delta w)_{tan} = 0$ on $\partial B$, we conclude

$$\frac{\partial f_I}{\partial x_m}(x) = 0$$

for all $I$ with $m \in I$. Now it is clear that

$$\frac{\partial |w|^2}{\partial x_m}(x) = 0.$$

We also observe that the following Neumann Sobolev type inequality holds on $M$

$$\inf_{a\in\mathbb{R}} \left( \int_B |f - a|^{\frac{2\nu}{\nu-2}} \right)^{\frac{\nu-2}{\nu}} \leq e^{C(1+\sqrt{K_p}r)}\, V^{-\frac{2}{\nu}}\, r^2 \int_B |\nabla f|^2$$

for all smooth functions $f$ on $B$.

From [**7**] lemma 2, we have

$$\left( \int_B |f|^{\frac{2\nu}{\nu-2}} \, dv \right)^{\frac{\nu-2}{\nu}} \leq e^{C(1+\sqrt{K_p}r)} V^{-2/\nu} r^2 \int_B |\nabla f|^2 \, dv + V^{-2/\nu} \int_B |f|^2 \, dv. \tag{3}$$

Let $\{w_i\}_{i=1}^k$ be the eigen $p$-forms satisfying the absolute boundary conditions with the corresponding non-zero eigenvalues $\{\lambda_i\}_{i=1}^k$ and we also assume $\{w_i\}_{i=1}^k$ are orthonormal and span $E$. If $w \in E$, then there exist $\{a_i\}_{i=1}^k$ such that $w = \sum_{i=1}^k a_i w_i$ , that is, $\Delta w = \sum_{i=1}^k \lambda_i a_i w_i$. By Bochner's formula

$$\frac{1}{2}\Delta |w|^2 \leq (\Delta w, w) - |\nabla w|^2 - p(m-p)K_p |w|^2$$

and the fact in [**7**] lemma 8

$$|\nabla |w||^2 \leq |\nabla w|^2$$

we have

$$\frac{1}{2}\Delta |w|^2 \leq (\Delta w, w) - |\nabla |w||^2 - p(m-p)K_p |w|^2 .$$

Let $\alpha \geq 1$, and we have

$$\frac{1}{2} \int_B |w|^{2\alpha-2} \Delta |w|^2 \leq \int_B |w|^{2\alpha-2} (\Delta w, w) - p(m-p)K_p \|w\|_{2\alpha}^{2\alpha} - \int_B |w|^{2\alpha-2} |\nabla |w||^2 . \tag{4}$$

Using the absolute boundary condition, the left hand side of inequality (4) becomes

$$\begin{aligned}\frac{1}{2}\int_B |w|^{2\alpha-2}\,\Delta\,|w|^2 =&(\alpha-1)\int_B |w|^{2\alpha-3}\,\langle\nabla\,|w|\,,\nabla\,|w|^2\rangle\\ =&2(\alpha-1)\int_B |w|^{2\alpha-2}\,\langle\nabla\,|w|\,,\nabla\,|w|\rangle\\ =&\frac{2(\alpha-1)}{\alpha^2}\int_B \langle\nabla\,|w|^{\alpha}\,,\nabla\,|w|^{\alpha}\rangle\end{aligned}$$

and the third term in the right hand side of (4) is

$$\int_B |w|^{2\alpha-2}\,|\nabla\,|w||^2 = \frac{1}{\alpha^2}\int_B |\nabla\,|w|^{\alpha}|^2\,.$$

Hence, (4) becomes

$$\begin{aligned}&\frac{2(\alpha-1)}{\alpha^2}\int_B |\nabla\,|w|^{\alpha}|^2\\ &\le \int_B |w|^{2\alpha-2}\,(\Delta w,w) - p(m-p)K_p\,\|w\|_{2\alpha}^{2\alpha} - \frac{1}{\alpha^2}\int_B |\nabla\,|w|^{\alpha}|^2\,.\end{aligned}$$

Let $f=|w|$, we have

$$\frac{2\alpha-1}{\alpha^2}\int_B |\nabla f^{\alpha}|^2 \le \int_B f^{2\alpha-2}\langle\Delta w,w\rangle - p(m-p)K_p\,\|f\|_{2\alpha}^{2\alpha}\,.$$

Applying (3) to the function $f^{\alpha}$

$$\left(\int_B |f^{\alpha}|^{2\beta}\,dv\right)^{1/\beta} \le e^{C(1+\sqrt{K_p}r)}V^{-2/\nu}r^2\int_B |\nabla f^{\alpha}|^2 + V^{-2/\nu}\int_B f^{2\alpha}$$

where $\beta=\frac{\nu}{\nu-2}$. We have

$$\begin{aligned}\|f\|_{2\alpha\beta}^{2\alpha} \le& \frac{\alpha^2}{2\alpha-1}e^{C(1+\sqrt{K_p}r)}V^{-2/\nu}r^2\bigg(\int_B f^{2\alpha-2}\langle\Delta w,w\rangle\\ &- p(m-p)K_p\,\|f\|_{2\alpha}^{2\alpha}\bigg) + V^{-2/\nu}\,\|f\|_{2\alpha}^{2\alpha}\\ \le& \frac{\alpha^2}{2\alpha-1}V^{-2/\nu}r^2\bigg(e^{C(1+\sqrt{K_p}r)}\bigg(\int_B f^{2\alpha-2}\langle\Delta w,w\rangle\\ &- p(m-p)K_p\,\|f\|_{2\alpha}^{2\alpha}\bigg) + r^{-2}\,\|f\|_{2\alpha}^{2\alpha}\bigg).\end{aligned}$$

Let $\alpha=\beta^i$, $i=0,1,2,\ldots$. Then,

$$\begin{aligned}\text{(5)}\qquad \|f\|_{2\beta^{i+1}}^{2\alpha} \le& \frac{\alpha^2}{2\alpha-1}V^{-2/\nu}r^2\bigg(e^{C(1+\sqrt{K_p}r)}\bigg(\int_B f^{2\beta^i-2}\langle\Delta w,w\rangle\\ &- p(m-p)K_p\,\|f\|_{2\beta^i}^{2\beta^i}\bigg) + r^{-2}\,\|f\|_{2\beta^i}^{2\beta^i}\bigg).\end{aligned}$$

When $i=0$, we have

$$\|f\|_{2\beta}^2 \le V^{-2/\nu} r^2 \bigg( e^{C(1+\sqrt{K_p}r)} \bigg( \int_B \langle \Delta w, w\rangle - p(m-p)K_p \|f\|_2^2 \bigg) + r^{-2} \|f\|_2^2 \bigg).$$

Since

$$\begin{aligned} \int_B \langle \Delta w, w\rangle &= \int_B \langle \lambda_i a_i w_i, a_j w_j\rangle \\ &= \lambda_i a_i^2 \le \lambda_k a_i^2 \\ &= \lambda_k \int_B \langle w, w\rangle. \end{aligned}$$

This implies

$$\|f\|_{2\beta}^2 \le V^{-2/\nu} r^2 \left( e^{C(1+\sqrt{K_p}r)} \left(\lambda_k - p(m-p)K_p\right) + r^{-2} \right) \|f\|_2^2 .$$

By Hölder inequality

$$\|f\|_2^2 \le V^{(\beta-1)/\beta} \|f\|_{2\beta}^2 .$$

If we let $\lambda_k^* = \lambda_k - p(m-p)K_p$,

$$V^{-(\beta-1)/\beta} \|f\|_2^2 \le V^{-2/\nu} r^2 \left( e^{C(1+\sqrt{K_p}r)} \lambda_k^* + r^{-2} \right) \|f\|_2^2 .$$

We claim that for $1 \le i < \infty$,

$$\begin{aligned} &V^{-(\beta-1)/\alpha\beta} \|f\|_{2\alpha}^2 \\ &\le \prod_{j=0}^{i} \left( \frac{2\beta^{2j}}{2\beta^j - 1} \right)^{\beta^{-j}} \bigg( V^{-2/\nu} r^2 \bigg( e^{C(1+\sqrt{K_p}r)} \lambda_k^* \\ &\qquad + r^{-2} \bigg) \bigg)^{\sum_{j=0}^{i} \beta^{-j}} \|f\|_2^2 . \end{aligned} \tag{6}$$

Assuming this is true for $\alpha = \beta^j$, $j = 0, \ldots, i-1$, by induction, we need to show (6) for $i$. Suppose the function $g = |\overline{w}|$, where $\overline{w} \in E$ with the property that for all $w \in E$,

$$\frac{\|g\|_{2\alpha}}{\|g\|_2} \ge \frac{\|w\|_{2\alpha}}{\|w\|_2} \qquad \text{for all } w \in E. \tag{7}$$

Without loss of the generality, we may use the the scaling and assume $\|g\|_2 = 1$. Then equation (5) becomes

$$\begin{aligned}\|g\|_{2\alpha\beta}^{2\alpha} \leq \frac{2\alpha^2}{2\alpha-1} V^{-2/\nu} r^2 \Big( & e^{C(1+\sqrt{K_p}r)} \Big( \int_B g^{2\alpha-2}(\Delta\overline{w}, \overline{w}) \\ & - p(m-p)K_p \|f\|_{2\alpha}^{2\alpha} \Big) + r^{-2} \|g\|_{2\alpha}^{2\alpha} \Big) \\ \leq \frac{2\alpha^2}{2\alpha-1} V^{-2/\nu} r^2 \Big( & e^{C(1+\sqrt{K_p}r)} (\|g\|_{2\alpha}^{2\alpha-1} \|\Delta\overline{w}\|_{2\alpha} \\ & - p(m-p)K_p \|f\|_{2\alpha}^{2\alpha}) + r^{-2} \|g\|_{2\alpha}^{2\alpha} \Big).\end{aligned}$$

We also note that if $s \geq 2$, then there exists a subset $\{\eta\} \subset \{1, 2, \ldots, k\}$ such that

$$\left\| \sum_{i=1}^k \lambda_i w_i \right\|_s \leq \left\| \sum_\eta \lambda_k w_\eta \right\|_s .$$

This property is proved in [**7**] lemma 17. Let $\overline{w} = \sum b_i w_i$, then $\Delta\overline{w} = \sum \lambda_i b_i w_i$ and we have

$$\begin{aligned}\|\Delta\overline{w}\|_{2\alpha} &= \left\| \sum_i \lambda_i b_i w_i \right\|_{2\alpha} \\ &\leq \left\| \sum_\eta \lambda_k b_\eta w_\eta \right\|_{2\alpha} \\ &= \lambda_k \left\| \sum_\eta b_\eta w_\eta \right\|_{2\alpha} \\ &\leq \lambda_k \|g\|_{2\alpha} \quad \left\| \sum_\eta b_\eta w_\eta \right\|_2 \quad \text{by (7)} \\ &\leq \lambda_k \|g\|_{2\alpha} .\end{aligned}$$

Thus,

$$\begin{aligned}\|g\|_{2\alpha\beta}^{2\alpha} &\leq \frac{2\alpha^2}{2\alpha-1} V^{-2/\nu} r^2 \left( e^{C(1+\sqrt{K_p}r)} \lambda_k^* \|g\|_{2\alpha}^{2\alpha} + r^{-2} \|g\|_{2\alpha}^{2\alpha} \right) \\ &= \frac{2\alpha^2}{2\alpha-1} V^{-2/\nu} r^2 \left( e^{C(1+\sqrt{K_p}r)} \lambda_k^* + r^{-2} \right) \|g\|_{2\alpha}^{2\alpha} .\end{aligned}$$

Using Moser iteration, we get the following result

$$\begin{aligned}\|g\|_{2\alpha\beta}^2 \leq & \prod_{j=0}^i \left( \frac{2\beta^{2j}}{2\beta^j - 1} \right)^{\beta^{-j}} \\ & \cdot \left( V^{-2/\nu} r^2 \left( e^{C(1+\sqrt{K_p}r)} \lambda_k^* + r^{-2} \right) \right)^{\sum_{j=0}^i \beta^{-j}} \|g\|_2^2 .\end{aligned}$$

On the other hand, by Hölder inequality, we have

$$\|g\|_{2\alpha}^2 \le V^{(\beta-1)/\alpha\beta} \|g\|_{2\alpha\beta}^2 .$$

Therefore,

$$V^{-(\beta-1)/\alpha\beta} \|g\|_{2\alpha}^2 \le \prod_{j=0}^{i} \left(\frac{\beta^{2j}}{2\beta^j - 1}\right)^{\beta^{-j}} \Big(V^{-2/\nu} r^2 \Big(e^{C(1+\sqrt{K_p} r)} \lambda_k^* + r^{-2}\Big)\Big)^{\sum_{j=0}^{i} \beta^{-j}} \|g\|_2^2 .$$

By (7), we get

$$V^{-(\beta-1)/\alpha\beta} \|f\|_{2\beta^i}^2 \le \prod_{j=0}^{i} \left(\frac{\beta^{2j}}{2\beta^j - 1}\right)^{\beta^{-j}} \Big(V^{-2/\nu} r^2 \Big(e^{C(1+\sqrt{K_p} r)} \lambda_k^* + r^{-2}\Big)\Big)^{\sum_{j=0}^{i} \beta^{-j}} \|f\|_2^2 .$$

Letting $i \to \infty$, then $\|f\|_{2\beta^i}^2 \to \|f\|_\infty^2$, and

$$\prod_{j=0}^{\infty} \left(\frac{\beta^{2j}}{2\beta^j - 1}\right)^{\beta^{-j}} \le \exp\left(\frac{1}{\beta^{1/2} - 1}\right) = c_1(\nu),$$

$$\sum_{j=0}^{\infty} \beta^{-j} = \frac{1}{1 - \frac{1}{\beta}} = \frac{\beta}{\beta - 1} = \frac{\frac{\nu}{\nu-2}}{\frac{\nu}{\nu-2} - 1} = \frac{\nu}{2}.$$

Hence, we obtain

$$\|f\|_\infty^2 \le C(\nu) \left(V^{-2/\nu} r^2 \left(e^{C(1+\sqrt{K_p} r)} \lambda_k^* + r^{-2}\right)\right)^{\nu/2} \|f\|_2^2 .$$

This means, for all $w \in E$, $w$ satisfies

(8)
$$\|w\|_\infty^2 \le C(\nu) (e^{C(1+\sqrt{K_p} r)} V^{-2/\nu} r^2 (\lambda_k - p(m-p) K_p + r^{-2}))^{\nu/2} \|w\|_2^2 .$$

We note that the Hodge Laplace Beltrami operator $\Delta = d\delta + \delta d$ is non-negative and self-adjoint on $B$ under the absolute boundary condition. Hence, using the standard elliptic theory, if we assume $\lambda_1$ to be the first non-zero eigenvalue, we have the property

$$0 < \lambda_1 \le \lambda_2 \le \cdots \to \infty.$$

This means, given a constant $b > 0$, there exists $k_0$ large enough such that the $k$-th nonzero eigenvalue $\lambda_k \ge b^{-2}$ on $B = B_q(r)$.

We select $0 < b \le r$, this implies

$$\lambda_k \ge r^{-2}.$$

Hence, from (8), we have (2)

$$\|w\|_\infty^2 \le C(\nu)(e^{C(1+\sqrt{K_p}r)}V^{-2/\nu}r^2(2\lambda_k - p(m-p)K_p))^{\nu/2}\|w\|_2^2$$

for all $w$ belong to the space span by the eigen $p$-forms $E$ correspondent to first $k$-th non-zero eigenvalues. Using the dimension estimate (1), we get

$$\begin{aligned}\frac{k}{V}\|w\|_2^2 \le& \tbinom{m}{p}\|w\|_\infty^2\\ \le& C(\nu)\tbinom{m}{p}(e^{C(1+\sqrt{K_p}r)}V^{-2/\nu}r^2(2\lambda_k - p(m-p)K_p))^{\nu/2}\|w\|_2^2,\end{aligned}$$

that is,

$$\lambda_k \ge C(\nu)\tbinom{m}{p}^{-2/\nu}e^{-C(1+\sqrt{K_p}r)}k^{2/\nu}r^{-2} - \frac{1}{2}p(m-p)K_p \quad \text{for all } k \ge k_0.$$

We complete the proof. q.e.d.

## 3. Proof of the Main Result

In this section, we will prove our main result Theorem 1.1. We start by establishing some preliminary lemmas. The following elementary fact appears in Li and Wang [**10**].

**Lemma 3.1.** *Let $V$ be a $k$-dimensional subspace of a vector space $W$. Assume that $W$ is endowed with an inner production $L$ and a bilinear form $\Phi$. Then for any given linearly independent set of vectors $\{w_1, \dots, w_{k-1}\} \subset W$, there exists an orthonormal basis $\{v_1, \dots, v_k\}$ of $V$ with respect to $L$ such that $\Phi(v_i, w_j) = 0$ for all $1 \le j < i \le k$.*

Let $\lambda_i$ denote the $i$-th nonzero eigenvalue on $p$-forms on $B_q(r) \subset (M, g)$ satisfying the absolute boundary conditions on $\partial B_q(r)$. We have the following lemma.

**Lemma 3.2.** *Let $V$ be a $k$-dimensional subspace of $H_l^p(M)$. For any fixed number $\beta > 1$ and any subspace $Y$ of $V$, let $\mathrm{tr}_{L_{\beta r}} L_r(Y)$ denote the trace of the bilinear form $L_r$ with respect to the inner product $L_{\beta r}$ on $Y$. Then we have*

$$\min_{\dim Y = k-s} \mathrm{tr}_{L_{\beta r}} L_r(Y) \le \sum_{i=s+1}^{k} \frac{8}{(\beta-1)^2 r^2 \lambda_i(\beta r)},$$

*where the minimum is taken over all subspaces $Y$ in $V$ with $\dim Y = k - s$ and $s = \dim H_p(B_q(r))$.*

*Proof.* Let $\phi$ be a nonnegative function defined on $B_q(\beta r)$ satisfying these conditions:

$$\phi = 1 \text{ on } B_q(r),\ 0 \le \phi \le 1 \text{ on } B_q(\beta r),\ \phi = 0 \text{ on } \partial B_q(\beta r),$$

and

$$|\nabla\phi| \leq \frac{2}{(\beta-1)r}.$$

Observe that by the property of unique continuation, $V$ is a $k$-dimensional subspace because

$$V \subset L^2(B_q(\beta r), dv) \cap L^2(B_q(r), \phi dv).$$

Applying Lemma 3.1 with $\{w_1, \ldots, w_k\}$ as the eigen $p$ forms of $B_q(\beta r)$ corresponding to the nonzero eigenvalues

$$\{\lambda_1(\beta r), \ldots, \lambda_k(\beta r)\},$$

we get an orthonormal basis $\{v_1, \ldots, v_k\}$ of $V$ with respect to the inner product $L_{\beta r}$. Hence

$$\Phi_{\beta r}(v_i, w_j) = \int_{B_q(\beta r)} (v_i, w_j)\phi dv = 0$$

for $1 \leq j < i \leq k$. Thus, for any $1 \leq i \leq k$, let

$$\begin{aligned}
&|v_i|^2 \equiv \langle v_i, v_i\rangle,\\
&\|v_i\|^2 = (v_i, v_i) = \int \langle v_i, v_i\rangle \quad \text{and}\\
&\mathrm{sgn} = (-1)^{m(p+1)+1}.
\end{aligned}$$

We have

$$\begin{aligned}
&\lambda_i(\beta r) \int_{B_q(\beta r)} |\phi v_i|^2\, dv\\
&\leq (d(\phi v_i), d(\phi v_i))_{L^2(B_q(\beta r))} + (\delta(\phi v_i), \delta(\phi v_i))_{L^2(B_q(\beta r))}\\
&= (d\phi \wedge v_i + \phi dv_i, d\phi \wedge v_i + \phi dv_i)_{L^2(B_q(\beta r))}\\
&\quad + (\phi\delta v_i + \mathrm{sgn} * (d\phi \wedge *v_i), \phi\delta v_i + \mathrm{sgn} * (d\phi \wedge *v_i))_{L^2(B_q(\beta r))}\\
&= \|d\phi \wedge v_i\|^2_{L^2(B_q(\beta r))} + 2(\phi dv_i, d\phi \wedge v_i)_{L^2(B_q(\beta r))}\\
&\quad + \|\phi dv_i\|^2_{L^2(B_q(\beta r))} + \|\phi\delta v_i\|^2_{L^2(B_q(\beta r))}\\
&\quad + 2(\phi\delta v_i, \mathrm{sgn} * (d\phi \wedge *v_i))_{L^2(B_q(\beta r))} + \|d\phi \wedge *v_i\|^2_{L^2(B_q(\beta r))}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
0 &= \int_{B_q(\beta r)} \phi^2 \langle v_i, \Delta v_i \rangle dv \\
&= (\phi^2 v_i, \Delta v_i)_{L^2(B_q(\beta r))} \\
&= (\delta(\phi^2 v_i), \delta v_i)_{L^2(B_q(\beta r))} + (d(\phi^2 v_i), dv_i)_{L^2(B_q(\beta r))} \\
&\quad + (\operatorname{ext}(N^*_{B_q(\beta r)})(\phi^2 v_i), dv_i)_{L^2(\partial B_q(\beta r))} \\
&\quad - (\operatorname{int}(N^*_{B_q(\beta r)}) d(\phi^2 v_i), \delta v_i)_{L^2(\partial B_q(\beta r))} \\
&= (\phi \delta v_i, \phi \delta v_i)_{L^2(B_q(\beta r))} + 2(\phi \delta v_i, \operatorname{sgn} * (d\phi \wedge * v_i))_{L^2(B_q(\beta r))} \\
&\quad + (\phi dv_i, \phi dv_i)_{L^2(B_q(\beta r))} + 2(\phi dv_i, d\phi \wedge v_i)_{L^2(B_q(\beta r))}.
\end{aligned}$$

We also have

$$\begin{aligned}
&(d(\phi v_i), d(\phi v_i))_{L^2(B_q(\beta r))} + (\delta(\phi v_i), \delta(\phi v_i))_{L^2(B_q(\beta r))} \\
&= \|d\phi \wedge v_i\|^2_{L^2(B_q(\beta r))} + \|d\phi \wedge * v_i\|^2_{L^2(B_q(\beta r))} \\
&= \int_{B_q(\beta r)} |d\phi \wedge v_i|^2 + |d\phi \wedge * v_i|^2 dv \\
&\le \sup |\nabla \phi|^2 \cdot \|v_i\|^2_{L^2(B_q(\beta r))} \\
&\le \frac{8}{(\beta-1)^2 r^2} \|v_i\|^2_{L^2(B_q(\beta r))} \\
&= \frac{8}{(\beta-1)^2 r^2},
\end{aligned}$$

since $v_i$ is orthonormal on $B_q(\beta r)$. Therefore, we get

$$\begin{aligned}
\int_{B_q(r)} |v_i|^2 \, dv &\le \int_{B_q(\beta r)} |\phi v_i|^2 \, dv \\
&\le \lambda_i^{-1}(\beta r) \Big\{ (d(\phi v_i), d(\phi v_i))_{L^2(B_q(\beta r))} \\
&\qquad + (\delta(\phi v_i), \delta(\phi v_i))_{L^2(B_q(\beta r))} \Big\} \\
&\le \frac{8}{(\beta-1)^2 r^2 \lambda_i(\beta r)}.
\end{aligned}$$

Hence, if we let $Y$ represent the space spanned by $\{v_{s+1}, \ldots, v_k\}$, then

$$\dim Y = k - s,$$

and

$$\operatorname{tr}_{L_{\beta r}} L_r(Y) = \sum_{i=s+1}^{k} \int_{B_q(r)} |v_i|^2 \, dv \le \sum_{i=s+1}^{k} \frac{8}{(\beta-1)^2 r^2 \lambda_i(\beta r)},$$

completing the proof. q.e.d.

The next result is due to Li [**8**].

**Lemma 3.3.** *Let $K$ be a $k$-dimensional linear space of $p$-forms defined on a manifold $N$ with polynomial volume growth of degree $m$. Suppose each $u \in K$ is of polynomial growth of at most degree $l$. For any $\beta > 1$, $\theta > 0$, and $r_0 > 0$, there exists $r > r_0$ such that if $\{u_i\}_{i=1}^k$ is an orthonormal basis of $K$ with respect to the inner product $L_{\beta r}(u, v) = \int_{B_q(\beta r)} \langle u, v \rangle$, then*

$$\sum_{i=1}^{k} \int_{B_q(r)} |u_i|^2 \geq k\beta^{-(2l+m+\theta)}.$$

We are now ready to prove Theorem 1.1 which is restated here.

**Theorem 3.4.** *Let $(M^m, g)$ be a complete Riemannian manifold with $K_p = 0$ on $(M, g)$. Then for any uniformly equivalent metric $g'$ on $M$ and for all $l \geq 1$, the space $H_l^p(M, g')$ is finite dimensional and its dimension satisfies the inequality*

$$\dim H_l^p(M, g') \leq C\, l^\nu$$

*for some constants $C > 0$ and $\nu > 2$.*

*Proof.* For any $k$-dimensional subspace $V$ of $H_l^p(M)$, by Lemma 3.3, where we set $\beta = 1 + \frac{1}{l}$ and $\delta = 1$, there exists $R > 0$ such that

$$C\, k \leq \mathrm{tr}_{L_{\beta R}} L_R(V).$$

Let $\lambda_k$ be the $k$-th eigenvalue of the Hodge Laplacian acting on $p$-forms on $B_q(R)$ satisfying the absolute boundary conditions on $\partial B_q(R)$ under the metric $g'$. Then by Lemma 2.1 and Proposition 2.2, we have

$$\lambda_k \geq C\, k^{2/\nu} \cdot R^{-2},$$

for all $k > \dim H_p(B_q(R))$. Combining with Lemma 3.2, we find there exists a subspace $Y$ in $V$ with

$$\dim Y = \dim V - \dim H_p(B_q(R))$$

so that

$$\begin{aligned} C\, k &\leq \mathrm{tr}_{L_{\beta R}} L_R(V) \\ &\leq \mathrm{tr}_{L_{\beta R}} L_R(Y) + \dim H_p(B_q(R)) \\ &\leq C\, l^2\, k^{1-2/\nu}. \end{aligned}$$

We conclude that $k \leq C\, l^\nu$ by using Lemma 2.1 and Proposition 2.2 on the estimate of $\dim H_p(B_q(R))$. Since $V$ is arbitrary, we have proved that

$$\dim H_l^p(M, g') \leq C\, l^\nu$$

for all $l \geq 1$. q.e.d.

## References

[1] M. Avellaneda & F.H. Lin, *Une thèorème de Liouville pour des èquations elliptique à coefficients pèriodiques*, Compt. Redus Acad. Sci. Paris **309** 1989 245–250, MR 1010728, Zbl 0691.35022.

[2] S.Y. Cheng & S.-T. Yau, *Differential equations on Riemannian manifolds and their geometric applications*, Comm. Pure Appl. Math. **28** 1975 333–354, MR 0385749, Zbl 0312.53031.

[3] J. Dódziuk, *Eigenvalues of the Laplacian on forms*, Proc. Amer. Math Soc. **85** 1982 438–443, MR 0656119, Zbl 0502.58038.

[4] J. Dódziuk & J. McGowan, *The spectrum of the Hodge Laplacian for a degenerating family of hyperbolic three manifolds*, Trans. Amer. Math Soc. **347** 1995 1981–1995, MR 1308007, Zbl 0849.58069.

[5] P. Li, *Lecture Notes on Geometric Analysis*, Lecture Notes Series, **6**, Research Institute of Mathematics and Global Analysis Research Center, Seoul National University, Seoul, 1993, MR 1320504, Zbl 0822.58001.

[6] ______, *Curvature and function theory on Riemannian manifolds*, Survey in Differential Geometry "In Honor of Atiyah, Bott, Hirzebruch, and Singer", International Press, Cambridge **VII** 2000 71–111, MR 1919432.

[7] ______, *On The Sobolev constant and the p-Spectrum of a compact Riemannian Manifold*, Ann. scient. Ec. Norm. Sup. **13** 1980 451–469, MR 0608289, Zbl 0466.53023.

[8] ______, *Harmonic sections of polynomial growth*, Math. Research Letters **4** 1997 35–44, MR 1432808, Zbl 0880.53039.

[9] P. Li & J. Wang, *Mean value inequalities*, Indiana Univ. Math. J. **48** 1999 1257–1283, MR 1757075, Zbl 1003.58026.

[10] ______, *Counting Massive sets and Dimensions of Harmonic functions*, J. Differential Geom. **53** 1999 237–278, MR 1802723, Zbl 1039.53036.

[11] L. Saloff-Coste, *Uniformly Elliptic operators on Riemannian manifolds*, J. Differential Geom. **36** 1992 417–450, MR 1180389, Zbl 0735.58032.

[12] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** 1975 201–228, MR 0431040, Zbl 0291.31002.

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